Robust Mixed H_2/H_∞ Control of 2-Dimensional Systems in Roesser Model [1]

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Abstract: The H_2 performance specification for one-dimensional systems has been known to be analytically and practically meaningful and widely used in the analysis and design of control and filtering systems. However, it is still of little use in the analysis and design of two-dimensional systems due to the structural and dynamical complexity of two-dimensional systems. In this paper, we extend the classical definition of the H_2 performance to two-dimensional systems and present a sufficient condition for evaluation of the H_2 performance of two-dimensional systems in Roesser model. Using this condition and the existing bounded real lemma for two-dimensional systems, we develop systematic design methods for mixed H_2/H_∞ and robust H_2/H_∞ control of two-dimensional systems with polytopic uncertainty. It is worth pointing out that our robust control approach can also be applied to give a solution to the dynamic output feedback control of one-dimensional systems with polytopic uncertainty which has not been solved in existing literature.

Keywords: H_2 control, H_∞ control, Optimal control, Two-dimensional systems.

1 Introduction

Many processes in practical applications are two-dimensional systems which exist in sound, seismic and underwater signal propagation, visual recognition of robotic systems, thermal processes, etc. Early work on control of 2-D systems can be found in [5, 6, 14]. Recently, there are a number of results reported on control and filtering of 2-D systems. Among these, [1, 2] developed a systematic approach to the solutions of H_∞ and robust control and filtering of 2-D systems in different system models. The results in [1, 2] are based on a bounded real lemma for 2-D systems and the solutions are computed using a linear matrix inequality (LMI) technique. In [13] a solution for mixed H_2/H_∞ filtering of 2-D systems is presented using the LMI technique, where a generalized H_2 norm is defined as the square of the peak amplitude of the output when the total energy of the past inputs is no greater than unity. Parallel to 1-D optimal control, the 2-D linear quadratic regulator (LQR) problem is developed in [7], where the design is based on the solution of several canonical equations of a Hamilitionian function. In [12], the quadratic optimal control of a discretized 2-D plant is reformulated into a 1-D optimal control problem which is restricted to finite 2-D indices.

H_2 optimal control for continuous and discrete time one-dimensional (1-D) systems is a classical problem in linear system theory. The objective of the H_2 control is to minimize the error energy of the system when the system is subject to unit impulse input or, equivalently, a white noise input of unit variance. Because of this analytically and practically meaningful specification, the H_2 problem and solution has been well studied and applied for several decades. However, the 1-D optimal control problem and its solution has not been systematically extended to the optimal control of 2-D systems, because of the structural and dynamical complexity of 2-D systems which significantly differ from 1-D systems. So far the existing result of optimal control of 2-D systems either used a different definition of the problem, such as the generalized H_2 norm in [13], or restricted the range of the 2-D system solution, such as the 1-D equivalent optimal control solution in [12].

In this paper, we consider linear discrete 2-D systems in Roesser model [10]. We will extend the definition of the H_2 performance specification for 1-D systems to 2-D systems and derive a sufficient condition for evaluation of the 2-D system H_2 performance. This condition is not necessary due to the difficulty that so far there has not been a necessary and sufficient condition similar to that given by the 1-D Lyapunov equation for the stability of 2-D systems in the state space [8, 9].

Using the condition for the 2-D system H_2 performance and the existing bounded real lemma [1, 2] for the 2-D

[1] Work supported by Australian Research Council

system H_∞ performance, we develop a systematic procedure for the design of the mixed H_2/H_∞ controller for 2-D systems in terms of LMIs. We further extend the design to robust H_2/H_∞ control which can cope with a general class of systems with polytopic parameter uncertainties. It should be noted that the problem of output feedback control of systems with polytopic uncertainty has not been solved even for 1-D systems due to the difficulty of obtaining a fixed controller from the solution of the associated LMIs. In this paper, we present a methodology to overcome this difficulty. Our solutions to the mixed H_2/H_∞ and robust H_2/H_∞ controllers can be efficiently computed from a set of LMIs or parameterized LMIs. The practical application in heat exchanger process shows the feasibility of the mixed H_2/H_∞ control and robust H_2/H_∞ control methods developed in this paper.

2 H_2 norm of 2-D systems in Roesser model

Let Z^+ be the set of nonnegative integers. A 2-D signal s with $s(i,j) \in R^n$, $i,j \in Z^+$ is said to belong to the 2-D ℓ_2-space if

$$\|s\|_2 = \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} s^T(i,j)s(i,j)} < \infty, \qquad (1)$$

where $\|\cdot\|_2$ denotes the ℓ_2 norm of s.

Roesser model for a 2-D system $G : u \mapsto y$ is defined by the following state equation

$$\begin{aligned} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= A \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Bu(i,j) \\ y(i,j) &= C \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Du(i,j) \end{aligned} \qquad (2)$$

where $x^h \in R^{n_h}, x^v \in R^{n_v}, u \in R^m$ and $y \in R^l$ are, respectively, the horizontal state, vertical state, input and the output of the system, A, B, C and D are the system matrices with appropriate dimension.

Let $E_k \in R^m$, $1 \le k \le m$ denote the kth column of the $m \times m$ identity matrix and δ be the 2-D discrete time unit impulse signal satisfying

$$\delta(i,j) = \begin{cases} 1, & if\ i=0, j=0; \\ 0, & otherwise \end{cases}$$

Further let the impulse response of the 2-D system G, subject to the zero boundary condition $x^h(-1,j) = 0, x^v(i,-1) = 0, x^h(0,0) = 0, x^v(0,0) = 0, i,j \ge 0$ and the input $u = E_k\delta$, $1 \le k \le m$, be

$$g_k = GE_k\delta.$$

If the 2-D system G is stable, its impulse response $g_k \in \ell_2$, for $1 \le k \le m$. We can follow the standard definition of the H_2 norm for one-dimensional (1-D) systems to define the H_2 norm of the 2-D system G as

$$\|G\|_2 = \sqrt{\sum_{k=1}^{m} \|g_k\|_2^2} = \sqrt{\sum_{k=1}^{m}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g_k^T(i,j)g_k(i,j)}. \qquad (3)$$

Physically, the H_2 norm of the system represents the amount of the system output energy when it is subject to the unit impulse input or a Gaussian white noise input with unit variance.

Consider that the 2-D system G in the Roesser model (2) is subject to the input $u = E_k\delta$, $1 \le k \le m$. Under such an input, let $x_k^h(i,j)$ and $x_k^v(i,j)$ denote the system horizontal state and vertical state, respectively. Further, let

$$x_k(i,j) = \begin{bmatrix} x_k^h(i,j) \\ x_k^v(i,j) \end{bmatrix}, \quad x_k^1(i,j) = \begin{bmatrix} x_k^h(i+1,j) \\ x_k^v(i,j+1) \end{bmatrix},$$

and write the matrices $B \in R^{(n_h+n_v)\times m}$ and $D \in R^{l\times m}$ as $B = [\tilde{B}_1 \quad \tilde{B}_2 \quad \cdots \quad \tilde{B}_m]$ and $D = [\tilde{D}_1 \quad \tilde{D}_2 \quad \cdots \quad \tilde{D}_m]$. Under the zero boundary condition, we can express the system impulse response g_k, $1 \le k \le m$, as

$$\begin{aligned} x_k^1(i,j) &= Ax_k(i,j) + BE_k\delta(i,j) \\ &= \begin{cases} \tilde{B}_k, & if\ i=0,\ j=0; \\ Ax_k(i,j), & otherwise. \end{cases} \\ g_k(i,j) &= Cx_k(i,j) + DE_k\delta(i,j) \\ &= \begin{cases} \tilde{D}_k & if\ i=0,\ j=0; \\ Cx_k(i,j) & otherwise. \end{cases} \end{aligned} \qquad (4)$$

We now present a sufficient condition for evaluation of the H_2 norm of the 2-D system in the Roessor model (2).

Theorem 1 *Given a positive scalar γ, the H_2 norm of the 2-D system G in the form (2) with zero boundary condition is bounded by γ, i.e. $\|G\|_2 < \gamma$, if there exists a block-diagonal matrix $P = diag\{P_h, P_v\} > 0$ such that*

$$A^TPA + C^TC - P < 0, \qquad (5)$$

and

$$trace(B^TPB + D^TD) - \gamma^2 < 0. \qquad (6)$$

Proof. Using the matrix $P = diag\{P_h, P_v\} > 0$, we introduce, for $1 \le k \le m$,

$$\Delta V_k(i,j) = x_k^1(i,j)^TPx_k^1(i,j) - x_k(i,j)^TPx_k(i,j). \qquad (7)$$

The existence of a diagonal positive definite solution P for (5) implies that the 2-D system (2) is stable [2] and that $x_k \in \ell_2$. This together with (7) further implies

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Delta V_k(i,j) = 0. \qquad (8)$$

On the other hand, we can use (4) to obtain

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta V_k(i,j) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_k(i,j)(A^TPA-P)x_k(i,j) + \tilde{B}_k^T P \tilde{B}_k \tag{9}$$

Using (8), (9) and (4), we can express the H_2 norm of the 2-D system G as

$$\begin{aligned} \|G\|_2^2 &= \sum_{k=1}^{m}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g_k^T(i,j)g_k(i,j) \\ &= \sum_{k=1}^{m}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_k^T(i,j)(A^TPA-P+C^TC)x_k(i,j) \\ &+ \ trace(B^TPB+D^TD). \end{aligned} \tag{10}$$

It follows immediately that $\|G\|_2 < \gamma$ if (5) and (6) are satisfied. □

We now introduce the definition of the H_∞ norm of the 2-D system G, denoted by $\|G\|_\infty$, as

$$\|G\|_\infty = \sup_{\|u\|_2 \le 1} \|Gu\|_2,$$

under the system zero boundary condition. A bounded real lemma for evaluation of the H_∞ norm of the 2-D system in the Roesser model (2) is presented in [2], which is stated as follows.

Lemma 1 *Given a positive scalar γ, the 2-D system (2) with zero boundary condition has an H_∞ noise attenuation γ if there exists a block-diagonal matrix $P = \{P_h, P_v\} > 0$, where $P_h \in R^{n_h \times n_h}$ and $P_v \in R^{n_v \times n_v}$ such that*

$$\begin{bmatrix} -P & PA & PB & 0 \\ A^TP & -P & 0 & C^T \\ B^TP & 0 & -\gamma^2 I & D^T \\ 0 & C & D & -I \end{bmatrix} < 0.$$

3 Mixed H_2/H_∞ control problems for 2-D systems in Roesser model

Consider a 2-D plant in the following Roesser model

$$\begin{aligned} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= A \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + B_1 w_1(i,j) \\ &\quad + B_2 w_2(i,j) + B_3 u(i,j) \\ z(i,j) &= C_1 \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + D_{11} w_1(i,j) \\ &\quad + D_{12} w_2(i,j) + D_{13} u(i,j) \\ y(i,j) &= C_2 \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \\ &\quad + D_{21} w_1(i,j) + D_{22} w_2(i,j) \end{aligned} \tag{11}$$

where $x^h \in R^{n_h}, x^v \in R^{n_v}, u \in R^m$ and $y \in R^l$ are, respectively, the horizontal state, vertical state, control input and measurement output of the plant, $w_1 \in R^{m_1}$ is a disturbance signal of bounded spectrum, $w_2 \in R^{m_2}$ is a signal of bounded power, $z \in R^p$ is the controlled output, A, B_1, B_2, B_3, C_1, C_2, D_{11}, D_{12}, D_{13}, D_{21} and D_{22} are real constant matrices of the plant of appropriate dimension.

Introduce the 2-D output feedback controller for the plant in the following Roesser model.

$$\begin{aligned} \begin{bmatrix} x_c^h(i+1,j) \\ x_c^v(i,j+1) \end{bmatrix} &= A_C \begin{bmatrix} x_c^h(i,j) \\ x_c^v(i,j) \end{bmatrix} + B_C y(i,j) \\ u(i,j) &= C_C \begin{bmatrix} x_c^h(i,j) \\ x_c^v(i,j) \end{bmatrix} + D_C y(i,j) \end{aligned} \tag{12}$$

where $x_c^h \in R^{n_h}, x_c^v \in R^{n_v}$ are, respectively, the horizontal state and vertical state of the controller, $A_C \in R^{(n_h+n_v)\times(n_h+n_v)}, B_C \in R^{(n_h+n_v)\times l}, C_C \in R^{m\times(n_h+n_v)}$ and $D_C \in R^{m\times l}$ are real constant matrices of the controller. Note that we consider a full order controller.

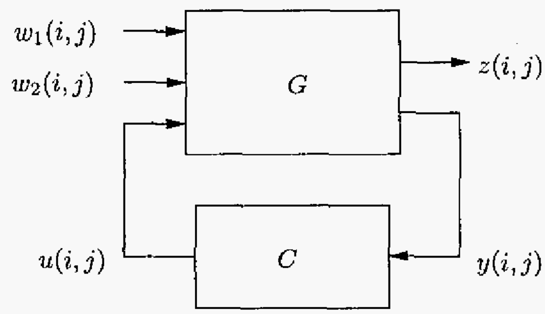

Figure 1: The closed-loop control system

Let $T : \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto z$ denote the closed loop system of the plant (11) and controller (12). The structure of the closed loop system is shown in Figure 1 and it can be written in the following Roesser model.

$$\begin{aligned} \begin{bmatrix} \bar{x}^h(i+1,j) \\ \bar{x}^v(i,j+1) \end{bmatrix} &= \bar{A} \begin{bmatrix} \bar{x}^h(i,j) \\ \bar{x}^v(i,j) \end{bmatrix} + \bar{B}_1 w_1(i,j) + \bar{B}_2 w_2(i,j) \\ z(i,j) &= \bar{C} \begin{bmatrix} \bar{x}^h(i,j) \\ \bar{x}^v(i,j) \end{bmatrix} + \bar{D}_1 w_1(i,j) + \bar{D}_2 w_2(i,j) \end{aligned} \tag{13}$$

where $\bar{x}^h(i,j) = \begin{bmatrix} x^h(i,j) \\ x_c^h(i,j) \end{bmatrix}$ and $\bar{x}^v(i,j) = \begin{bmatrix} x^v(i,j) \\ x_c^v(i,j) \end{bmatrix}$ are the horizontal state and vertical state of the closed loop system, respectively, and

$$\bar{A} = \Pi \begin{bmatrix} A + B_3 D_C C_2 & B_3 C_C \\ B_C C_2 & A_C \end{bmatrix} \Pi^T,$$

$$\bar{B}_1 = \Pi \begin{bmatrix} B_1 + B_3 D_C D_{21} \\ B_C D_{21} \end{bmatrix}, \quad \bar{B}_2 = \Pi \begin{bmatrix} B_2 + B_3 D_C D_{22} \\ B_C D_{22} \end{bmatrix},$$

$$\begin{aligned} \bar{D}_1 &= D_{11} + D_{13} D_C D_{21}, \\ \bar{D}_2 &= D_{12} + D_{13} D_C D_{22}, \\ \bar{C} &= \begin{bmatrix} C_1 + D_{13} D_C C_2 & D_{13} C_C \end{bmatrix} \Pi^T, \end{aligned} \quad \Pi = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Let $T_1 : w_1 \mapsto z$ denote the closed loop system subject to the spectrum bounded disturbance input w_1 with $w_2 = 0$ and $T_2 : w_2 \mapsto z$ denote the closed loop system subject to the energy bounded disturbance input w_2 with $w_1 = 0$. Then we state the 2-D mixed H_2/H_∞ control problem as: for given constants $\gamma_1, \gamma_2 > 0$, find, if exists, a 2-D output feedback controller of the form (12) for the 2-D plant (11) such that the closed-loop system is stable and has mixed specified H_2 performance $\|T_1\|_2 < \gamma_1$ and H_∞ performance $\|T_2\|_\infty < \gamma_2$.

Now consider that the 2-D plant in the Roesser model (11) is subject to polytopic parameter uncertainties such that its system matrices are unknown but are known to belong to the following n-polytopic convex polyhedron.

$$\begin{aligned} \mathcal{M} &= \{(A, B_1, B_2, B_3, C_1, C_2, D_{11}, D_{12}, D_{13}, D_{21}, D_{22}) \\ &= \sum_{i=1}^{n} \xi_i \Big(A^{(i)}, B_1^{(i)}, B_2^{(i)}, B_3^{(i)}, C_1^{(i)}, C_2^{(i)}, D_{11}^{(i)}, \\ &\quad D_{12}^{(i)}, D_{13}^{(i)}, D_{21}^{(i)}, D_{22}^{(i)}\Big), \sum_{i=1}^{n} \xi_i = 1, \xi_i \geq 0 \Big\}, \end{aligned} \tag{14}$$

where $A^{(i)}, B_1^{(i)}, B_2^{(i)}, B_3^{(i)}, C_1^{(i)}, C_2^{(i)}, D_{11}^{(i)}, D_{12}^{(i)}, D_{13}^{(i)}, D_{21}^{(i)}, D_{22}^{(i)}, 1 \leq i \leq n$, are known matrices of appropriate dimension and ξ_i, $1 \leq i \leq n$, are free nonnegative parameters constrained by $\sum_{i=1}^{n} \xi_i = 1$.

When the parameters of the uncertain 2-D plant in the Roesser model (11) belong to the polyhedron (14), we can state the 2-D robust H_2/H_∞ control problem as: for given constants $\gamma_1, \gamma_2 > 0$, find, if exists, a 2-D output feedback controller of the form (12) for the 2-D uncertain plant (11) such that the closed-loop system has mixed specified H_2 performance $\|T_1\|_2 < \gamma_1$ and H_∞ performance $\|T_2\|_\infty < \gamma_2$ for any parameter uncertainty from the polyhedron (14).

Remark 1 *The mixed H_2/H_∞ control of 1-D systems without parameter uncertainty has been studied in, for example, [3]. In the presence of polytopic type of parameter uncertainty, the robust H_2 or H_∞ filtering has been investigated in [4, 11]. However, the problem of dynamic output feedback control of systems with parameter uncertainties of polytopic type remains open even for 1-D systems. In this paper, we present an approach to address this problem for 2-D systems with polytopic uncertainty which is also applicable to 1-D systems.*

4 Mixed H_2/H_∞ control of 2-D systems in Roesser model

Using the result of Theorem 1 and the bounded real lemma of [2], we present a solution for the mixed H_2/H_∞ 2-D control problem in the following theorem.

Theorem 2 *The 2-D mixed H_2/H_∞ control problem for the plant (11) is solvable if there exist matrices $S > 0$, Θ, Λ, Γ, D_C and block-diagonal matrices $X = diag\{X^h, X^v\}$, $Y = diag\{Y^h, Y^v\}$, $N = diag\{N^h, N^v\}$, $H_{11} = diag\{H_{11}^h, H_{11}^v\} > 0$, $H_{22} = diag\{H_{22}^h, H_{22}^v\} > 0$, $H_{12} = diag\{H_{12}^h, H_{12}^v\}$, $K_{11} = diag\{K_{11}^h, K_{11}^v\} > 0$, $K_{22} = diag\{K_{22}^h, K_{22}^v\} > 0$, $K_{12} = diag\{K_{12}^h, K_{12}^v\}$ such that*

$$\begin{bmatrix} H_{11} & \star & \star & \star & \star \\ H_{12}^T & H_{22} & \star & \star & \star \\ XA + \Gamma C_2 & \Theta & X + X^T - H_{11} & \star & \star \\ A + B_3 D_C C_2 & AY^T + B_3\Lambda & I + N^T - H_{12}^T & Y + Y^T - H_{22} & \star \\ C_1 + D_{13} D_C C_2 & C_1 Y^T + D_{13}\Lambda & 0 & 0 & I \end{bmatrix} > 0 \tag{15}$$

$$\begin{bmatrix} S & \star & \star & \star \\ XB_1 + \Gamma D_{21} & X + X^T - H_{11} & \star & \star \\ B_1 + B_3 D_C D_{21} & I + N^T - H_{12}^T & Y + Y^T - H_{22} & \star \\ D_{11} + D_{13} D_C D_{21} & 0 & 0 & I \end{bmatrix} > 0 \tag{16}$$

$$\begin{bmatrix} -K_{11} & \star & \star & \star & \star & \star \\ -K_{12}^T & -K_{22} & \star & \star & \star & \star \\ XA + \Gamma C_2 & \Theta & -X - X^T + K_{11} & \star & \star & \star \\ A + B_3 D_C C_2 & AY^T + B_3\Lambda & -I - N^T + K_{12}^T & -Y - Y^T + K_{22} & \star & \star \\ 0 & 0 & (XB_2 + \Gamma D_{22})^T & (B_2 + B_3 D_C D_{22})^T & -\gamma_2^2 I & \star \\ C_1 + D_{13} D_C C_2 & C_1 Y^T + D_{13}\Lambda & 0 & 0 & D_{12} + D_{13} D_C D_{22} & -I \end{bmatrix} < 0, \tag{17}$$

$$trace(S) < \gamma_1^2, \tag{18}$$

where $\star$ represents any arbitrary real block entries of the matrices with appropriate dimension.

Proof. Following from Theorem 1 and the bounded real lemma of [2] and using the slack variable technique [3], the mixed H_2/H_∞ performance can be met if there exist a matrix $S > 0$, block-diagonal matrices $\tilde{P} = diag\{\tilde{P}_h, \tilde{P}_v\}$, $P = diag\{P_h, P_v\} > 0$ and $\bar{P} = diag\{\bar{P}_h, \bar{P}_v\} > 0$, all with appropriate dimension, such that

$$\begin{bmatrix} P & \bar{A}^T \tilde{P}^T & \bar{C}^T \\ \tilde{P}\bar{A} & \tilde{P} + \tilde{P}^T - P & 0 \\ \bar{C} & 0 & I \end{bmatrix} > 0, \tag{19}$$

$$\begin{bmatrix} S & \bar{B}_1^T\tilde{P}^T & \bar{D}_1^T \\ \tilde{P}\bar{B}_1 & \tilde{P}+\tilde{P}^T-P & 0 \\ \bar{D}_1 & 0 & I \end{bmatrix} > 0, \quad (20)$$

$$trace(S) < \gamma_1^2, \quad (21)$$

$$\begin{bmatrix} -\bar{P} & \bar{A}^T\tilde{P}^T & 0 & \bar{C}^T \\ \tilde{P}\bar{A} & -\tilde{P}-\tilde{P}^T+\bar{P} & \tilde{P}\bar{B}_2 & 0 \\ 0 & \bar{B}_2^T\tilde{P}^T & -\gamma_2^2 I & \bar{D}_2^T \\ \bar{C} & 0 & \bar{D}_2 & -I \end{bmatrix} < 0. \quad (22)$$

Obviously, $\Pi = \Pi^{-1} = \Pi^T$. Pre- and post-multiplying (19) by $diag\{\Pi, \Pi, I\}$, (20) by $diag\{I, \Pi, I\}$ and (22) by $diag\{\Pi, \Pi, I, I\}$, respectively, the matrix inequalities (19), (20) and (22) are equivalent to

$$\begin{bmatrix} Q & \Pi\bar{A}^T\Pi\tilde{Q}^T & \Pi\bar{C}^T \\ \tilde{Q}\Pi\bar{A}\Pi & \tilde{Q}+\tilde{Q}^T-Q & 0 \\ \bar{C}\Pi & 0 & I \end{bmatrix} > 0, \quad (23)$$

$$\begin{bmatrix} S & \bar{B}_1^T\Pi\tilde{Q}^T & \bar{D}_1^T \\ \tilde{Q}\Pi\bar{B}_1 & \tilde{Q}+\tilde{Q}^T-Q & 0 \\ \bar{D}_1 & 0 & I \end{bmatrix} > 0, \quad (24)$$

$$\begin{bmatrix} -\bar{Q} & \Pi\bar{A}^T\Pi\tilde{Q}^T & 0 & \Pi\bar{C}^T \\ \tilde{Q}\Pi\bar{A}\Pi & -\tilde{Q}-\tilde{Q}^T+\bar{Q} & \tilde{Q}\Pi\bar{B}_2 & 0 \\ 0 & \bar{B}_2^T\Pi\tilde{Q}^T & -\gamma_2^2 I & \bar{D}_2^T \\ \bar{C}\Pi & 0 & \bar{D}_2 & -I \end{bmatrix} < 0 \quad (25)$$

where $Q = \Pi P \Pi$, $\bar{Q} = \Pi\bar{P}\Pi$ and $\tilde{Q} = \Pi\tilde{P}\Pi$.

Observe that $\tilde{Q}+\tilde{Q}^T > Q > 0$. Hence, $\tilde{Q}$ is invertible. Denote $\tilde{Q} = \begin{bmatrix} X & U \\ U_1 & \star \end{bmatrix}$, $\tilde{Q}^{-1} = \begin{bmatrix} Y & V \\ V_1 & \star \end{bmatrix}$ and $Z = \begin{bmatrix} I & 0 \\ Y & V \end{bmatrix}$, so we have

$$Z\tilde{Q}Z^T = \begin{bmatrix} X & U \\ I & 0 \end{bmatrix} Z^T = \begin{bmatrix} X & N \\ I & Y^T \end{bmatrix},$$

where $N = XY^T + UV^T$. Note that the matrices X U, Y and V are all diagonal matrices. Pre-multiply (23), (24) and (25) by $diag\{Z, Z, I\}$, $diag\{I, Z, I\}$ and $diag\{Z, Z, I, I\}$, respectively. Then post-multiply (23), (24) and (25) by $diag\{Z^T, Z^T, I\}$, $diag\{I, Z^T, I\}$ and $diag\{Z^T, Z^T, I, I\}$, respectively. We can obtain the result of the theorem by letting $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix} = ZQZ^T > 0$, $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} = Z\bar{Q}Z^T > 0$, $\Theta = XAY^T + XB_3D_CC_2Y^T + UB_CC_2Y^T + XB_3C_CV^T + UA_CV^T$, $\Gamma = XB_3D_C + UB_C$ and $\Lambda = D_CC_2Y^T + C_CV^T$. □

If the LMIs (18) exist a solution, it is easy to see that

$$\begin{bmatrix} X+X^T & I+N \\ I+N^T & Y+Y^T \end{bmatrix} > 0.$$

Multiply the above from the left by $[-Y \quad I]$ and from the right by $[-Y \quad I]^T$. We obtain that

$$Y(XY^T - N) + (YX^T - N^T)Y^T > 0.$$

It is then clear that $N - XY^T$ is invertible. Since $N - XY^T = UV^T$, U and V are also invertible. Thus, a solution to the controller in the form (12) can be obtained from the solutions to the LMIs (15)-(18), where $C_C = (\Lambda - D_CC_2Y^T)V^{-T}$, $B_C = U^{-1}(\Gamma - XB_3D_C)$, $A_C = U^{-1}[\Theta - X(A + B_3D_CC_2)Y^T - XB_3C_CV^T - UB_CC_2Y^T]V^{-T}$ and the diagonal matrices U, V can be chosen arbitrarily such that $XY^T + UV^T = N$ is satisfied.

5 Robust H_2/H_∞ control of Roesser model

To find a solution for the robust H_2/H_∞ 2-D control problem when the plant is subject to the polytopic parameter uncertainties as modelled in (14), we first introduce the following useful technical result.

Lemma 2 *Let $\Delta \in R^{n\times n} > 0$, $\Psi \in R^{m\times m} > 0$, $W_2 \in R^{n\times n}$ and $W_1 \in R^{n\times n}$ are nonsingular, $\Xi \in R^{n\times n}$, $\Omega_1 \in R^{m\times n}$, $\Omega_2 \in R^{m\times n}$. There exists matrix $H > 0$ such that*

$$\begin{bmatrix} W_1HW_1^T & \Xi^T & \Omega_1^T \\ \Xi & \Delta - W_2HW_2^T & \Omega_2^T \\ \Omega_1 & \Omega_2 & \Psi \end{bmatrix} > 0 \quad (26)$$

if there exist scalar $\varepsilon > 0$ and matrices $\bar{H} > 0$, $\hat{H} > 0$ such that

$$\begin{bmatrix} \bar{H} & \Xi^T & \Omega_1^T \\ \Xi & \Delta - \hat{H} & \Omega_2^T \\ \Omega_1 & \Omega_2 & \Psi \end{bmatrix} > 0, \quad (27)$$

$$\begin{bmatrix} 2\varepsilon I - \varepsilon^2\bar{H} & W_1^{-T}W_2^T \\ W_2W_1^{-1} & \hat{H} \end{bmatrix} > 0. \quad (28)$$

Proof. Let $\hat{H} = W_2HW_2^T$, so we know that (27) can lead to (26) if

$$\bar{H} < W_1HW_1^T = W_1W_2^{-1}\hat{H}W_2^{-T}W_1^T. \quad (29)$$

It is easy to know that (29) holds if and only if

$$W_1^{-1}\bar{H}W_1^{-T} < W_2^{-1}\hat{H}W_2^{-T}$$

or

$$\bar{H}^{-1} > W_1^{-T}W_2^T\hat{H}^{-1}W_2W_1^{-1}.$$

Since $(\bar{H}^{-1} - \varepsilon I)^T\bar{H}(\bar{H}^{-1} - \varepsilon I) \geq 0$, we have

$$\bar{H}^{-1} \geq 2\varepsilon I - \varepsilon^2\bar{H}.$$

Therefore, we know (29) holds if $2\varepsilon I - \varepsilon^2\bar{H} > W_1^{-T}W_2^T\hat{H}^{-1}W_2W_1^{-1}$, which is equivalent to (28). This completes the proof. □

Using the above lemma, the result of Theorem 1 and the bounded real lemma of [2], we present a solution for the robust H_2/H_∞ 2-D control problem in the following theorem.

Theorem 3 *The robust H_2/H_∞ 2-D control problem for the plant (11) with the parameter uncertainty (14) is solvable if, for some scalars $\varepsilon_1^{(i,j)} > 0$, $\varepsilon_2^{(i,j)} > 0$ and all $1 \le i \le n$, $1 \le j \le n$, there exist matrices $S > 0$, Θ, Λ, Γ, D_C and block-diagonal matrices*

$$
\begin{aligned}
\bar{X} &= diag\left\{\bar{X}^h, \bar{X}^v\right\}, \bar{H}_{11}^{(i,j)} = diag\left\{\bar{H}_{11}^{(i,j)h}, \bar{H}_{11}^{(i,j)v}\right\} > 0,\\
Y &= diag\left\{Y^h, Y^v\right\}, \bar{H}_{22}^{(i,j)} = diag\left\{\bar{H}_{22}^{(i,j)h}, \bar{H}_{22}^{(i,j)v}\right\} > 0,\\
\bar{N} &= diag\left\{\bar{N}^h, \bar{N}^v\right\}, \hat{H}_{11}^{(i,j)} = diag\left\{\hat{H}_{11}^{(i,j)h}, \hat{H}_{11}^{(i,j)v}\right\} > 0,\\
\bar{H}_{12}^{(i,j)} &= diag\left\{\bar{H}_{12}^{(i,j)h}, \bar{H}_{12}^{(i,j)v}\right\},\\
\hat{H}_{12}^{(i,j)} &= diag\left\{\hat{H}_{12}^{(i,j)h}, \hat{H}_{12}^{(i,j)v}\right\},\\
\bar{K}_{12}^{(i,j)} &= diag\left\{\bar{K}_{12}^{(i,j)h}, \bar{K}_{12}^{(i,j)v}\right\},\\
\hat{K}_{12}^{(i,j)} &= diag\left\{\hat{K}_{12}^{(i,j)h}, \hat{K}_{12}^{(i,j)v}\right\},\\
\hat{H}_{22}^{(i,j)} &= diag\left\{\hat{H}_{22}^{(i,j)h}, \hat{H}_{22}^{(i,j)v}\right\} > 0\\
\bar{K}_{11}^{(i,j)} &= diag\left\{\bar{K}_{11}^{(i,j)h}, \bar{K}_{11}^{(i,j)v}\right\} > 0\\
\bar{K}_{22}^{(i,j)} &= diag\left\{\bar{K}_{22}^{(i,j)h}, \bar{K}_{22}^{(i,j)v}\right\} > 0\\
\hat{K}_{11}^{(i,j)} &= diag\left\{\hat{K}_{11}^{(i,j)h}, \hat{K}_{11}^{(i,j)v}\right\} > 0\\
\hat{K}_{22}^{(i,j)} &= diag\left\{\hat{K}_{22}^{(i,j)h}, \hat{K}_{22}^{(i,j)v}\right\} > 0
\end{aligned}
$$

all with appropriate dimension, such that

$$
\begin{bmatrix}
\hat{H}_{11}^{(i,j)} & \star & \star & \star & \star\\
\bar{H}_{12}^{(i,j)T} & \hat{H}_{22}^{(i,j)} & \star & \star & \star\\
A^{(i)} + B_3^{(j)} D_C C_2^{(i)} + \Gamma C_2^{(i)} & A^{(i)} + B_3^{(j)} D_C C_2^{(i)} + \Gamma C_2^{(i)} + B_3^{(j)}\Lambda + \Theta & \bar{X} + \bar{X}^T - \hat{H}_{11}^{(i,j)} & \star & \star\\
A^{(i)} + B_3^{(j)} D_C C_2^{(i)} & A^{(i)} + B_3^{(j)} (D_C C_2^{(i)} + \Lambda) & \bar{X}^T + \bar{N}^T - \hat{H}_{12}^{(i,j)T} & Y + Y^T - \hat{H}_{22}^{(i,j)} & \star\\
C_1^{(i)} + D_{13}^{(j)} D_C C_2^{(i)} & C_1^{(i)} + D_{13}^{(j)} (D_C C_2^{(i)} + \Lambda) & 0 & 0 & I
\end{bmatrix} > 0, \tag{30}
$$

$$
\begin{bmatrix}
-\bar{K}_{11}^{(i,j)} & \star & \star & \star & \star & \star\\
-\bar{K}_{12}^{(i,j)T} & -\bar{K}_{22}^{(i,j)} & \star & \star & \star & \star\\
A^{(i)} + B_3^{(j)} D_C C_2^{(i)} + \Gamma C_2^{(i)} & A^{(i)} + B_3^{(j)} D_C C_2^{(i)} + \Gamma C_2^{(i)} + B_3^{(j)}\Lambda + \Theta & -\bar{X}^T - \bar{N}^T + \hat{K}_{12}^{(i,j)T} & \star & \star & \star\\
A^{(i)} + B_3^{(j)} D_C C_2^{(i)} & A^{(i)} + B_3^{(j)} (D_C C_2^{(i)} + \Lambda) & (B_2^{(i)} + B_3^{(j)} D_C D_{22}^{(i)} + \Gamma D_{22}^{(i)})^T & -Y - Y^T + \hat{K}_{22}^{(i,j)} & \star & \star\\
0 & 0 & -\bar{X} - \bar{X}^T + \hat{K}_{11}^{(i,j)} & (B_2^{(i)} + B_3^{(j)} D_C D_{22}^{(i)})^T & -\gamma_2^2 I & \star\\
C_1^{(i)} + D_{13}^{(j)} D_C C_2^{(i)} & C_1^{(i)} + D_{13}^{(j)} (D_C C_2^{(i)} + \Lambda) & 0 & 0 & D_{12}^{(i)} + D_{13}^{(j)} D_C D_{22}^{(i)} & -I
\end{bmatrix} < 0, \tag{31}
$$

$$
\begin{bmatrix}
S & \star & \star & \star\\
B_1^{(i)} + B_3^{(j)} D_C D_{21}^{(i)} + \Gamma D_{21}^{(i)} & \bar{X} + \bar{X}^T - \hat{H}_{11}^{(i,j)} & \star & \star\\
B_1^{(i)} + B_3^{(j)} D_C D_{21}^{(i)} & \bar{X}^T + \bar{N}^T - \hat{H}_{12}^{(i,j)T} & Y + Y^T - \hat{H}_{22}^{(i,j)} & \star\\
D_{11}^{(i)} + D_{13}^{(j)} D_C D_{21}^{(i)} & 0 & 0 & I
\end{bmatrix} > 0 \tag{32}
$$

$$
trace(S) < \gamma_1^2 \tag{33}
$$

$$
\begin{bmatrix}
2\varepsilon_1^{(i,j)} I - (\varepsilon_1^{(i,j)})^2 \bar{H}_{11}^{(i,j)} & -(\varepsilon_1^{(i,j)})^2 \bar{H}_{12}^{(i,j)} & \bar{X}^T & 0\\
-(\varepsilon_1^{(i,j)})^2 \bar{H}_{12}^{(i,j)T} & 2\varepsilon_1^{(i,j)} I - (\varepsilon_1^{(i,j)})^2 \bar{H}_{22}^{(i,j)} & 0 & Y^T\\
\bar{X} & 0 & \hat{H}_{11}^{(i,j)} & \hat{H}_{12}^{(i,j)}\\
0 & Y & \hat{H}_{12}^{(i,j)T} & \hat{H}_{22}^{(i,j)}
\end{bmatrix} > 0 \tag{34}
$$

and

$$
\begin{bmatrix}
2\varepsilon_2^{(i,j)} I - (\varepsilon_2^{(i,j)})^2 \bar{K}_{11}^{(i,j)} & -(\varepsilon_2^{(i,j)})^2 \bar{K}_{12}^{(i,j)} & \bar{X}^T & 0\\
-(\varepsilon_2^{(i,j)})^2 \bar{K}_{12}^{(i,j)T} & 2\varepsilon_2^{(i,j)} I - (\varepsilon_2^{(i,j)})^2 \bar{K}_{22}^{(i,j)} & 0 & Y^T\\
\bar{X} & 0 & \hat{K}_{11}^{(i,j)} & \hat{K}_{12}^{(i,j)}\\
0 & Y & \hat{K}_{12}^{(i,j)T} & \hat{K}_{22}^{(i,j)}
\end{bmatrix} > 0. \tag{35}
$$

Proof. Following from Theorem 1 and the bounded real lemma of [2] and using the slack variable technique, the robust mixed H_2/H_∞ performance can be met if there exist a matrix $S > 0$, block-diagonal matrices $P^{(i,j)} = diag\left\{P_h^{(i,j)}, P_v^{(i,j)}\right\} > 0$, $\bar{P}^{(i,j)} = diag\left\{\bar{P}_h^{(i,j)}, \bar{P}_v^{(i,j)}\right\} > 0$ and $\tilde{P} = diag\left\{\tilde{P}_h, \tilde{P}_v\right\}$, for all $1 \le i \le n, 1 \le j \le n$ and with appropriate dimension, such that

$$
\begin{bmatrix}
P^{(i,j)} & \bar{A}^{(i,j)T}\tilde{P}^T & \bar{C}^{(i,j)T}\\
\tilde{P}\bar{A}^{(i,j)} & \tilde{P} + \tilde{P}^T - P^{(i,j)} & 0\\
\bar{C}^{(i,j)} & 0 & I
\end{bmatrix} > 0, \tag{36}
$$

$$
\begin{bmatrix}
S & \bar{B}_1^{(i,j)T}\tilde{P}^T & \bar{D}_1^{(i,j)T}\\
\tilde{P}\bar{B}_1^{(i,j)} & \tilde{P} + \tilde{P}^T - P^{(i,j)} & 0\\
\bar{D}_1^{(i,j)} & 0 & I
\end{bmatrix} > 0, \tag{37}
$$

$$
trace(S) < \gamma_1^2, \tag{38}
$$

$$
\begin{bmatrix}
-\bar{P}^{(i,j)} & \bar{A}^{(i,j)T}\tilde{P}^T & 0 & \bar{C}^{(i,j)T}\\
\tilde{P}\bar{A}^{(i,j)} & -\tilde{P} - \tilde{P}^T + \bar{P}^{(i,j)} & \tilde{P}\bar{B}_2^{(i,j)} & 0\\
0 & \bar{B}_2^{(i,j)T}\tilde{P}^T & -\gamma_2^2 I & \bar{D}_2^{(i,j)T}\\
\bar{C}^{(i,j)} & 0 & \bar{D}_2^{(i,j)} & -I
\end{bmatrix} < 0 \tag{39}
$$

where

$$
\begin{aligned}
\bar{A}^{(i,j)} &= \Pi \begin{bmatrix} A^{(i)} + B_3^{(j)} D_C C_2^{(i)} & B_3^{(j)} C_C\\ B_C C_2^{(i)} & A_C \end{bmatrix} \Pi^T,\\
\bar{B}_1^{(i,j)} &= \Pi \begin{bmatrix} B_1^{(i)} + B_3^{(j)} D_C D_{21}^{(i)}\\ B_C D_{21}^{(i)} \end{bmatrix},\\
\bar{B}_2^{(i,j)} &= \Pi \begin{bmatrix} B_2^{(i)} + B_3^{(j)} D_C D_{22}^{(i)}\\ B_C D_{22}^{(i)} \end{bmatrix},\\
\bar{C}^{(i,j)} &= \begin{bmatrix} C_1^{(i)} + D_{13}^{(j)} D_C C_2^{(i)} & D_{13}^{(j)} C_C \end{bmatrix} \Pi^T,\\
\bar{D}_1^{(i,j)} &= D_{11}^{(i)} + D_{13}^{(j)} D_C D_{21}^{(i)}, \bar{D}_2^{(i,j)} = D_{12}^{(i)} + D_{13}^{(j)} D_C D_{22}^{(i)}.
\end{aligned}
$$

Pre- and post-multiplying (36) by $diag\{\Pi, \Pi, I\}$, (37) by $diag\{I, \Pi, I\}$ and (39) by $diag\{\Pi, \Pi, I, I\}$, respectively, the matrix inequalities (36), (37) and (39) are equivalent to

$$\begin{bmatrix} Q^{(i,j)} & \Pi\bar{A}^{(i,j)T}\Pi\tilde{Q}^T & \Pi\bar{C}^{(i)T} \\ \tilde{Q}\Pi\bar{A}^{(i,j)}\Pi & \tilde{Q}+\tilde{Q}^T-Q^{(i,j)} & 0 \\ \bar{C}^{(i,j)}\Pi & 0 & I \end{bmatrix} > 0, \quad (40)$$

$$\begin{bmatrix} S & \bar{B}_1^{(i,j)T}\Pi\tilde{Q}^T & \bar{D}_1^{(i,j)T} \\ \tilde{Q}\Pi\bar{B}_1^{(i,j)} & \tilde{Q}+\tilde{Q}^T-Q^{(i,j)} & 0 \\ \bar{D}_1^{(i,j)} & 0 & I \end{bmatrix} > 0, \quad (41)$$

$$\begin{bmatrix} -\bar{Q}^{(i,j)} & \Pi\bar{A}^{(i,j)T}\Pi\tilde{Q}^T & 0 & \Pi\bar{C}^{(i,j)T} \\ \tilde{Q}\Pi\bar{A}^{(i,j)}\Pi & -\tilde{Q}-\tilde{Q}^T+\bar{Q}^{(i,j)} & \tilde{Q}\Pi\bar{B}_2^{(i,j)} & 0 \\ 0 & \bar{B}_2^{(i,j)T}\Pi\tilde{Q}^T & -\gamma_2^2 I & \bar{D}_2^{(i,j)T} \\ \bar{C}^{(i,j)}\Pi & 0 & \bar{D}_2^{(i,j)} & -I \end{bmatrix} < 0, \quad (42)$$

where $Q^{(i,j)} = \Pi P^{(i,j)}\Pi$, $\bar{Q}^{(i,j)} = \Pi\bar{P}^{(i,j)}\Pi$ and $\tilde{Q} = \Pi\tilde{P}\Pi$.

It can be easily seen that $\tilde{Q}$ is non-singular. Denote $\tilde{Q} = \begin{bmatrix} X & U \\ U_1 & \star \end{bmatrix}$, $\tilde{Q}^{-1} = \begin{bmatrix} Y & V \\ V_1 & \star \end{bmatrix}$ and $Z = \begin{bmatrix} I & 0 \\ Y & V \end{bmatrix}$. Note that the matrices X U, Y and V are all diagonal matrices. Further let, for all $1 \le i \le n$ and $1 \le j \le n$,

$$H^{(i,j)} = \begin{bmatrix} H_{11}^{(i,j)} & H_{12}^{(i,j)} \\ H_{12}^{(i,j)T} & H_{22}^{(i,j)} \end{bmatrix} = ZQ^{(i,j)}Z^T,$$

$$K^{(i,j)} = \begin{bmatrix} K_{11}^{(i,j)} & K_{12}^{(i,j)} \\ K_{12}^{(i,j)T} & K_{22}^{(i,j)} \end{bmatrix} = Z\bar{Q}^{(i,j)}Z^T,$$

$$\Theta^{(i,j)} = XA^{(i)}Y^T + XB_3^{(j)}D_C C_2^{(i)}Y^T + UB_C C_2^{(i)} Y^T + XB_3^{(j)}C_C V^T + UA_C V^T,$$

$$\Gamma^{(i,j)} = XB_3^{(j)}D_C + UB_C,$$

$$\Lambda^{(i)} = D_C C_2^{(i)}Y^T + C_C V^T, N = XY^T + UV^T.$$

We can pre-multiply (40) by $diag\{Z, Z, I\}$, (41) by $diag\{I, Z, I\}$ and (42) $diag\{Z, Z, I, I\}$, respectively, and post-multiply (40) by $diag\{Z^T, Z^T, I\}$, (41) by $diag\{I, Z^T, I\}$ and (42) $diag\{Z^T, Z^T, I, I\}$, respectively, to obtain

$$\begin{bmatrix} H_{11}^{(i,j)} & \star & \star & \star & \star \\ H_{12}^{(i,j)T} & H_{22}^{(i,j)} & \star & \star & \star \\ XA^{(i)}+\Gamma^{(i,j)}C_2^{(i)} & \Theta^{(i,j)} & X+X^T-H_{11}^{(i,j)} & \star & \star \\ A^{(i)}+B_3^{(j)}D_C C_2^{(i)} & A^{(i)}Y^T+B_3^{(j)}\Lambda^{(i)} & I+N^T-H_{12}^{(i,j)T} & Y+Y^T-H_{22}^{(i,j)} & \star \\ C_1^{(i)}+D_{13}^{(j)}D_C C_2^{(i)} & C_1^{(i)}Y^T+D_{13}^{(j)}\Lambda^{(i)} & 0 & 0 & I \end{bmatrix} > 0, \quad (43)$$

$$\begin{bmatrix} S & \star & \star & \star \\ XB_1^{(i)}+\Gamma^{(i,j)}D_{21}^{(i)} & X+X^T-H_{11}^{(i,j)} & \star & \star \\ B_1^{(i)}+B_3^{(j)}D_C D_{21}^{(i)} & I+N^T-H_{12}^{(i,j)T} & Y+Y^T-H_{22}^{(i,j)} & \star \\ D_{11}^{(i)}+D_{13}^{(j)}D_C D_{21}^{(i)} & 0 & 0 & I \end{bmatrix} > 0 \quad (44)$$

$$\begin{bmatrix} -K_{11}^{(i,j)} & \star & \star & \star & \star & \star \\ -K_{12}^{(i,j)T} & -K_{22}^{(i,j)} & \star & \star & \star & \star \\ XA^{(i)}+\Gamma^{(i,j)}C_2^{(i)} & \Theta^{(i,j)} & -X-X^T+K_{11}^{(i,j)} & \star & \star & \star \\ A^{(i)}+B_3^{(j)}D_C C_2^{(i)} & A^{(i)}Y^T+B_3^{(j)}\Lambda^{(i)} & -I-N^T+K_{12}^{(i,j)T} & -Y-Y^T+K_{22}^{(i,j)} & \star & \star \\ 0 & 0 & (XB_2^{(i)}+\Gamma^{(i,j)}D_{22}^{(i)})^T & (B_2^{(i)}+B_3^{(j)}D_C D_{22}^{(i)})^T & -\gamma_2^2 I & \star \\ C_1^{(i)}+D_{13}^{(j)}D_C C_2^{(i)} & C_1^{(i)}Y^T+D_{13}^{(j)}\Lambda^{(i)} & 0 & 0 & D_{12}^{(i)}+D_{13}^{(j)}D_C D_{22}^{(i)} & -I \end{bmatrix} < 0. \quad (45)$$

Observe from that (43) that X and Y are invertible. Pre- and post-multiplying (43) by $diag\{I, Y^{-1}, X^{-1}, I, I\}$ and $diag\{I, Y^{-T}, X^{-T}, I, I\}$, respectively, denoting

$$\bar{H}^{(i,j)} = \begin{bmatrix} \bar{H}_{11}^{(i,j)} & \bar{H}_{12}^{(i,j)} \\ \bar{H}_{12}^{(i,j)T} & \bar{H}_{22}^{(i,j)} \end{bmatrix} = W_1 H^{(i,j)} W_1^T$$

$$\hat{H}^{(i,j)} = \begin{bmatrix} \hat{H}_{11}^{(i,j)} & \hat{H}_{12}^{(i,j)} \\ \hat{H}_{12}^{(i,j)T} & \hat{H}_{22}^{(i,j)} \end{bmatrix} = W_2 H^{(i,j)} W_2^T$$

$W_1 = \begin{bmatrix} I & 0 \\ 0 & Y^{-1} \end{bmatrix}$, $W_2 = \begin{bmatrix} X^{-1} & 0 \\ 0 & I \end{bmatrix}$, $\bar{N} = Y^T + X^{-1}UV^T$, $\bar{X} = X^{-1}$, $\Gamma = \bar{X}UB_C$, $\Lambda = C_C V^T Y^{-T}$ and $\Theta = \bar{X}UA_C V^T Y^{-T}$ and applying Lemma 2, yields, there exists $H^{(i,j)} > 0$ such that the above matrix inequality or equivalently (43) holds if there exist a scalar $\varepsilon_1^{(i,j)} > 0$, matrices $\bar{H}^{(i,j)} > 0$ and $\hat{H}^{(i,j)} > 0$ such that (30) and (34) hold.

Similarly, pre- and post-multiplying (44) by $diag\{I, X^{-1}, I, I\}$ and $diag\{I, X^{-T}, I, I\}$ and (45) by $diag\{I, Y^{-1}, X^{-1}, I, I, I\}$, and $diag\{I, Y^{-T}, X^{-T}, I, I, I\}$, respectively, denoting

$$\bar{K}^{(i,j)} = \begin{bmatrix} \bar{K}_{11}^{(i,j)} & \bar{K}_{12}^{(i,j)} \\ \bar{K}_{12}^{(i,j)T} & \bar{K}_{22}^{(i,j)} \end{bmatrix} = W_1 K^{(i,j)} W_1^T$$

$$\hat{K}^{(i,j)} = \begin{bmatrix} \hat{K}_{11}^{(i,j)} & \hat{K}_{12}^{(i,j)} \\ \hat{K}_{12}^{(i,j)T} & \hat{K}_{22}^{(i,j)} \end{bmatrix} = W_2 K^{(i,j)} W_2^T.$$

Similarly, by applying Lemma 2 again, the theorem is established. □

If the robust H_2/H_∞ 2-D control problem for the plant (11) with the polytopic parameter uncertainty as modelled in (14) is solvable, a controller in the Roesser model (12) can be obtained from the solutions to the LMIs (30)-(34), where $C_C = \Lambda Y^T V^{-T}$, $B_C = U^{-1}\bar{X}^{-1}\Gamma$, $A_C = U^{-1}\bar{X}^{-1}\Theta Y^T V^{-T}$ and the diagonal matrices U, V can be chosen arbitrarily as long as $Y^T + X^{-1}UV^T = \bar{N}$ is satisfied. Note that U and V are non-singular.

Remark 2 *As mentioned earlier, the dynamic output feedback control of systems with polytopic uncertainty has not been solved by existing literature even for 1-D systems. Theorem 3 presents a solution to this open problem. Note that the solution involves searching for appropriate scaling parameters $\varepsilon_1^{(i,j)}$ and $\varepsilon_2^{(i,j)}$ which is in general difficult although some optimization algorithms such as the* **fminsearch** *in Matlab Optimization Toolbox may be applied to obtain a local optimal solution. In practice, to simplify the search, one may set $\varepsilon_1^{(i,j)} = \varepsilon_1$ and $\varepsilon_2^{(i,j)} = \varepsilon_2$, $i, j = 1, 2, \cdots, n$ but at a cost of suboptimal performance.*

6 Example

6.1 Mixed H_2/H_∞ control of heat exchanger

Consider the following equation describing heat exchanger [5]

$$\frac{\partial T(x,t)}{\partial x} = -\frac{\partial T(x,t)}{\partial t} - T(x,t) + U(t) \quad (46)$$

where $T(x,t)$ is usually the temperature at space $x \in [0, x_f]$ and time $t \in [0, \infty]$. $U(t)$ is a given force function. Taking

$$T(i,j) = T(i\Delta x, j\Delta t), \quad U(j) = U(j\Delta t),$$

and $x^h(i,j) = T(i-1,j)$, $x^v(i,j) = T(i,j)$, from (46), we can obtain the following Roesser model

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} U(j) \tag{47}$$

where $a_1 = 1 - \frac{\Delta t}{\Delta x} - \Delta t$, $a_2 = \frac{\Delta t}{\Delta x}$ and $b = \Delta t$.

Let $\Delta t = 0.1$ and $\Delta x = 0.2$, we have $a_1 = 0.4$, $a_2 = 0.5$ and $b = 0.1$. If we take noise disturbance into account and assume that the whole system is modelled in the form of (11) with

$$A = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.4 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 \\ 0.08 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, C_1 = \begin{bmatrix} 0.4 & 0.3 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 10 \end{bmatrix},$$

and $D_{11} = 0.01, D_{12} = 0.01, D_{13} = 0.1, D_{21} = 0, D_{22} = 0.05, D_{23} = 0$.

Given $\gamma_1 = 0.059$ and $\gamma_2 = 0.05$, by Theorem 1, we obtain

$$A_C = \begin{bmatrix} -0.1002 & -0.0068 \\ 0.0271 & -0.0006 \end{bmatrix}, B_C = \begin{bmatrix} -0.0259 \\ -0.0008 \end{bmatrix},$$

$$C_C = \begin{bmatrix} 14.3184 & -0.7154 \end{bmatrix}, D_C = -0.3056.$$

We can thus have the state-space model of the close-loop system in the form of (13) and the corresponding transfer functions as follows:

$$\begin{aligned} T_{zw_1} &= \bar{C}(\bar{Z} - \bar{A})^{-1}\bar{B}_1 + \bar{D}_1 \\ T_{zw_2} &= \bar{C}(\bar{Z} - \bar{A})^{-1}\bar{B}_2 + \bar{D}_2, \end{aligned} \tag{48}$$

where $\bar{Z} = diag\{z_h I_2, z_v I_2\}$. In addition, Figure 2 and Figure 3 show the magnitude of frequency responses of the closed-loop systems $T_{zw_1}(e^{j\omega_h}, e^{j\omega_v})$ and $T_{zw_2}(e^{j\omega_h}, e^{j\omega_v})$, respectively, over all frequencies, where 1.0 corresponds to π. From Figure 3, it can be known that the maximum value of $T_{zw_2}(e^{j\omega_h}, e^{j\omega_v})$ is 0.0411 which is below the specified upper bound $\gamma_2 = 0.05$.

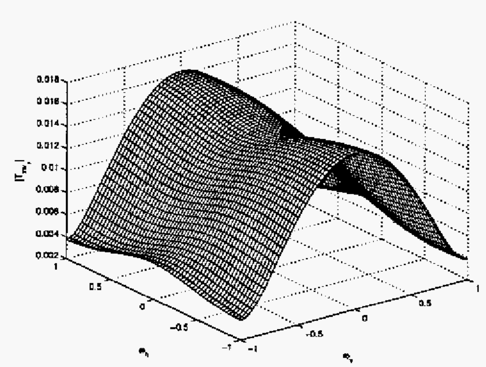

Figure 2: The frequency response of $T_{zw_1}(e^{j\omega_h}, e^{j\omega_v})$

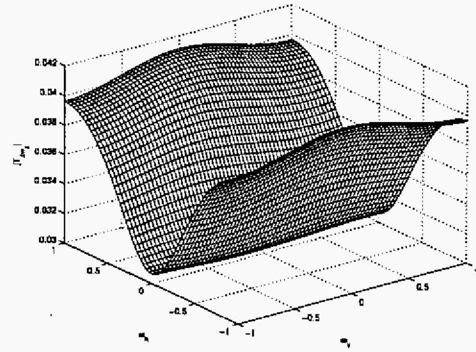

Figure 3: The frequency response of $T_{zw_2}(e^{j\omega_h}, e^{j\omega_v})$

6.2 Robust H_2/H_∞ control of heat exchanger

We further consider the case that the heat exchanger system (46) is subject to polytopic parameter uncertainties such that the system matrices of its Roesser model belong to the 2-polytopic convex polyhedron in the form of (14), where

$$A^{(1)} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.3 \end{bmatrix}, A^{(2)} = \begin{bmatrix} 0 & 1 \\ 0.25 & 0.4 \end{bmatrix}, B_1^{(1)} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

$$B_1^{(2)} = \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix}, B_2^{(1)} = \begin{bmatrix} 0.01 \\ 0.3 \end{bmatrix}, B_2^{(2)} = \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix},$$

$$B_3^{(1)} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, B_3^{(2)} = \begin{bmatrix} 0.32 \\ 0.16 \end{bmatrix}, C_1^{(1)} = \begin{bmatrix} 0.3 & 0.4 \end{bmatrix},$$

$$C_1^{(2)} = \begin{bmatrix} 0.1 & 0.5 \end{bmatrix}, C_2^{(1)} = \begin{bmatrix} 1 & 1 \end{bmatrix}, C_2^{(2)} = \begin{bmatrix} 2 & 0.5 \end{bmatrix},$$

$$D_{11}^{(1)} = 0, D_{11}^{(2)} = 0.1, D_{12}^{(1)} = -0.3, D_{12}^{(2)} = 0.1, D_{13}^{(1)} = 0.1,$$

$$D_{13}^{(2)} = 0.2, D_{21}^{(1)} = 0.1, D_{21}^{(2)} = 0.12, D_{22}^{(1)} = 0.1, D_{22}^{(2)} = 0.15.$$

Let $\varepsilon_1^{(1,1)} = 10$, $\varepsilon_1^{(1,2)} = 10^{-4}$, $\varepsilon_1^{(2,1)} = 10^{-4}$, $\varepsilon_1^{(2,2)} = 10^{-4}$, $\varepsilon_2^{(1,1)} = 5$, $\varepsilon_2^{(1,2)} = 10^{-4}$, $\varepsilon_2^{(2,1)} = 10^{-4}$, $\varepsilon_2^{(2,2)} = 10^{-4}$, $\gamma_1 = 0.1$ and $\gamma_2 = 0.38$. Following from Theorem 3, we have

$$\bar{X} = \begin{bmatrix} 13.4222 & 0 \\ 0 & 6.9470 \end{bmatrix}, Y = \begin{bmatrix} 18.0847 & 0 \\ 0 & 9.5279 \end{bmatrix},$$

$$\bar{N} = \begin{bmatrix} -3.7812 & 0 \\ 0 & -0.5433 \end{bmatrix}, \Theta = \begin{bmatrix} 0.2763 & -0.3362 \\ -0.1010 & 0.1217 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} -0.2592 \\ -0.0173 \end{bmatrix}, \Lambda = \begin{bmatrix} 0.3895 & -1.2903 \end{bmatrix}.$$

Then, choose $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so

$$V = U^{-1}\bar{X}^{-1}(\bar{N} - Y^T) = \begin{bmatrix} -1.6291 & 0 \\ 0 & -1.4497 \end{bmatrix}.$$

Therefore, we obtain the robust H_2/H_∞ controller of the form (12) with

$$A_C = \begin{bmatrix} -0.2285 & 0.1646 \\ 0.1614 & -0.1152 \end{bmatrix}, B_C = \begin{bmatrix} -0.0193 \\ -0.0025 \end{bmatrix},$$

$$C_C = \begin{bmatrix} -4.3236 & 8.4803 \end{bmatrix}, D_C = -3.2342.$$

Let the actual 2-D system be obtained from the above two-vertex polytope with $\xi_1 = 0.7$ and $\xi_2 = 0.3$. The frequency responses of T_{zw_1} and T_{zw_2} defined as (48) are shown in Figure 4 and Figure 5, respectively. We can find that the specified H_2 and H_∞ performances are met.

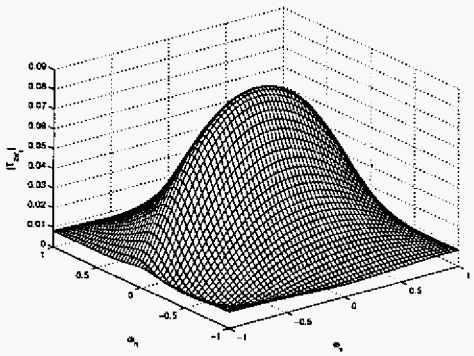

Figure 4: The frequency response of $T_{zw_1}(e^{j\omega_h}, e^{j\omega_v})$

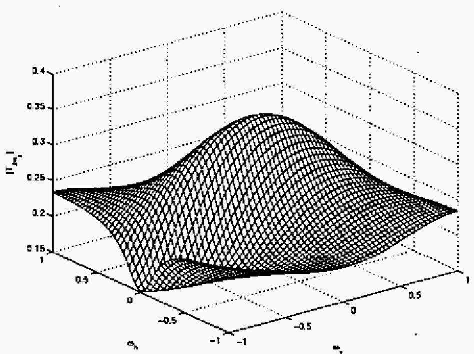

Figure 5: The frequency response of $T_{zw_2}(e^{j\omega_h}, e^{j\omega_v})$

7 Conclusion

In this paper, we extended the classical definition of the H_2 performance to 2-D systems and presented a sufficient condition for evaluation of the H_2 performance of 2-D system in Roesser model. Using this condition and the existing bounded real lemma for 2-D systems, we develop systematic design methods for mixed H_2/H_∞ and robust H_2/H_∞ control of 2-D systems in Roesser model. The solutions for the H_2/H_∞ control are in the form of LMIs which can be efficiently computed by existing software.

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